FIBER BUNDLE WITH NON-ZERO SIGNATURE

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For a fiber bundle of oriented closed manifolds

$$F \to E \to B,$$

when is $\mathrm{sd}(\pi) := \sigma(E) - \sigma(B)\sigma(F) = 0$? This was established by Chern, Hirzebruch, and Serre [CHS57] for bundles where the monodromy action of $\pi_1(B)$ on $H^*(F;\mathbb{Q})$ is trivial. Examples where $\mathrm{sd}(\pi) \neq 0$ were found by Atiyah, Hirzebruch, and Kodaira [Ati69; Hir69; Kod67]. Atiyah's example is in the case $\dim(F) = \dim(B) = 2$ where $\mathrm{sd}(\pi) = \sigma(E)$. Thus non-multiplicativity is finding such an E with non-zero signature. Atiyah constructs such a bundle using algebraic geometry.

The purpose of this short note is to construct an example with non-zero signature, using moduli spaces of manifolds. This is well-known to experts, see e.g. a similar note by Randal-Williams[RW15], but I felt it was worth making such an application more widely available.

Theorem 1. *There exists a closed, oriented manifold E of non-zero signature, fitting into a bundle of closed oriented surfaces*

$$\Sigma_g \longrightarrow E^4 \xrightarrow{\pi} B^2$$

Proof. By the Hirezebruch signature theorem $\sigma(E^4) = \langle \frac{p_1(E^4)}{3}, [E] \rangle$ so it suffices to construct an E with $p_1(E) \neq 0$.

Oriented surface bundles are classified by maps $B \to B\,\mathrm{Diff}^+(\Sigma_g)$. By Madsen-Weiss [MW07] we know that

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(B\,\mathrm{Diff}^+(\Sigma_g);\mathbb{Q})$$

is an isomorphism in degrees $* \leq \frac{g-3}{2}$, where the κ-classes (aka. Miller–Morita–Mumford-classes) have degrees $|\kappa_i| = 2i$. In general, the characteristic classes of the bundle are given by $\kappa_c(\pi) = \pi_!(c(T_vE)) \in H^{k-d}(B;\mathbb{Q})$ where T_vE is the vertical tangent bundle, and c is a monomial of degree k in the Pontryagin classes and d is the dimension of the manifold fiber. By the defining short exact sequence of vector bundles

$$T_vE \to TE \to \pi^*TB$$

and the sum formula for Pontryagin classes (noting that there is no contribution from π^*TB since B is 2-dimensional) we have $p_1(E) = p_1(T_vE)$.

Hence if $c = p_1$, then we just need an E with $\kappa_1(E) \neq 0$, as then $p_1 \neq 0$. In particular, we just need a classifying map $\Sigma_h \to B\,\mathrm{Diff}^+(\Sigma_g)$ which is an injection in $H^2(-;\mathbb{Q})$, or equivalently a surjection in rational homology. But every degree 2 homology class is represented by a map from a surface and since $H_2(B\,\mathrm{Diff}^+(\Sigma_g);\mathbb{Q})$ is 1-dimensional (when $g \geq 7$), we can just hit the generator. □

References

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